On certain definite integrals and infinite series*

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The definite integrals, which I want to discuss now, are connected very intimately to those infinite series, which I considered in the paper published in this journal on the hypergeometric series Volume 15, pp. 138 sq., which, to represent them more easily, I will denote by these functions:

$$1. \quad 1 + \frac{\alpha \cdot x}{\beta \cdot 1} + \frac{\alpha(\alpha+1) \cdot x^2}{\beta(\beta+1) \cdot 1 \cdot 2} + \frac{\alpha(\alpha+1)(\alpha+2) \cdot x^3}{\beta(\beta+1)(\beta+2) \cdot 1 \cdot 2 \cdot 3} + \cdots = \phi(\alpha, \beta, x)$$

$$2. \quad 1 + \frac{x}{\alpha \cdot 1} + \frac{x^2}{\alpha(\alpha+1) \cdot 1 \cdot 2} + \frac{x^3}{\alpha(\alpha+1)(\alpha+2) \cdot 1 \cdot 2 \cdot 3} + \cdots = \psi(\alpha, x)$$

$$3. \quad 1 - \frac{\alpha \cdot \beta}{1 \cdot x} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot x^2} - \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot x^3} + \cdots = \chi(\alpha, \beta, x)$$

Hence the transformation of these series found in the mentioned paper can be exhibited this way:

$$4. \quad \phi(\alpha, \beta, x) = e^x \phi(\beta - \alpha, \beta, -x)$$

$$5. \quad \psi(\alpha, x) = e^{\pm 2\sqrt{x}} \phi\left(\alpha - \tfrac{1}{2}, 2\alpha - 1, \pm 4\sqrt{x}\right),$$

which formula is the same as

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$$6. \quad \phi(\alpha, 2\alpha, x) = e^{\frac{x}{2}} \psi\left(\alpha + \frac{1}{2}, \frac{x^2}{16}\right)$$

and

$$7. \quad \chi(\alpha, \beta, x) = \frac{x^\alpha \Pi(\beta - \alpha - 1)}{\Pi(\beta - 1)} \phi(\alpha, \alpha - \beta, x) + \frac{x^\beta \Pi(\alpha - \beta - 1)}{\Pi(\alpha - 1)} \phi(\beta, \beta - \alpha + 1, x).$$

Having prepared these things, I will first discuss the question about the integral

$$8. \quad y = \int_0^\infty u^{\alpha-1} \cdot e^{-u} \cdot e^{-\frac{x}{u}} du,$$

from which it follows

$$\frac{dy}{dx} = -\int_0^\infty u^{\alpha-2} \cdot e^{-u} \cdot e^{-\frac{x}{u}} du, \quad \frac{d^2y}{dx^2} = \int_0^\infty u^{\alpha-3} \cdot e^{-u} \cdot e^{-\frac{x}{u}} du$$

by differentiation of the quantity $u^{\alpha-1} \cdot e^{-u} \cdot e^{-\frac{x}{u}}$:

$$d\left(u^{\alpha-1} \cdot e^{-u} \cdot e^{-\frac{x}{u}}\right)$$
$$= -u^{\alpha-1} \cdot e^{-u} \cdot e^{-\frac{x}{u}} du + (\alpha - 1) u^{\alpha-2} \cdot e^{-u} \cdot e^{-\frac{x}{u}} du + x \cdot u^{\alpha-3} \cdot e^{-u} \cdot e^{-\frac{x}{u}} du,$$

and by integration between the limits 0 and ∞

$$0 = -\int_0^\infty u^{\alpha-1} \cdot e^{-u} \cdot e^{-\frac{x}{u}} du + (\alpha - 1) \int_0^\infty u^{\alpha-2} \cdot e^{-u} \cdot e^{-\frac{x}{u}} du$$
$$+ x \int_0^\infty u^{\alpha-3} \cdot e^{-u} \cdot e^{-\frac{x}{u}} du$$

or, what is the same,

$$9. \quad 0 = y + (\alpha - 1) \frac{dy}{dx} - x \frac{d^2y}{dx^2}.$$

The complete integral of this equation can easily be found in terms of the series we denoted by the function ψ

$$10. \quad A \cdot \psi(1-\alpha, x) + B \cdot x^{\alpha} \cdot \psi(1+\alpha, x),$$

where A and B are arbitrary constants. Hence this expression of the propounded integral follows

$$\int_0^\infty u^{\alpha-1} \cdot e^{-u} \cdot e^{-\frac{x}{u}} du = A \cdot \psi(1-\alpha, x) + B \cdot x^{\alpha} \cdot \psi(1+\alpha, x).$$

The determination of the constant A is easy; for, if we assume α to be a positive quantity and put $x = 0$, we have

$$\int_0^\infty u^{\alpha-1} \cdot e^{-u} du = A$$

or

$$A = \Pi(\alpha - 1).$$

To determine the constant B in the same way, the integral y must be transformed by means of the substitution $u = \frac{x}{v}$, whence

$$\int_0^\infty u^{\alpha-1} \cdot e^{-u} \cdot e^{-\frac{x}{u}} du = x^{\alpha} \int_0^\infty v^{-\alpha-1} \cdot e^{-v} \cdot e^{-\frac{x}{v}} dv,$$

having applied this transformation of the integral, equation (11.) goes over into this one:

$$\int_0^\infty v^{-\alpha-1} \cdot e^{-v} \cdot e^{-\frac{x}{v}} dv = A \cdot x^{-\alpha} \psi(1-\alpha, x) + B \cdot \psi(1+\alpha, x),$$

hence, if we assume the quantity α to be negative and put $x = 0$, we have

$$\int_0^\infty v^{-\alpha-1} e^{-u} dv = B$$

or

$$B = \Pi(-\alpha - 1),$$

having substituted which values of the constants, we have:

$$12. \quad \int_0^\infty u^{\alpha-1} \cdot e^{-u} \cdot e^{-\frac{x}{u}} du = \Pi(\alpha-1)\psi(1-\alpha,x) + \Pi(-\alpha-1)x^{\alpha-1}x^{\alpha}\psi(1+\alpha,x).$$

But some doubts concerning the determination of the constants are to be removed, which can arise from the fact that the one constant was found for $\alpha > 0$, but the determination of the other constant demands the opposite assumption. But nevertheless it is clear that these conditions were superfluous, if in the determination of the constant we would not have used the value $x = 0$, but certain other positive values, and hence the values of the constants would not have been any different. Furthermore, it is to be noted that formula (12.) only holds, if x is a positive quantity, otherwise that integral would become infinite; but if x is positive, this integral has a finite value, whatever the quantity α is, positive or negative.
From this formula (12.) one can deduce another integral, which is expressed by two series of the form $\phi(\alpha, \beta, x)$. Writing xv instead of v, multiplying by $e^{-v} \cdot v^{\beta-1} \cdot dv$ and integrating from 0 to ∞, we find

$$\int_0^\infty \int_0^\infty u^{\alpha-1} \cdot e^{-u} \cdot v^{\beta-1} \cdot e^{-v} \cdot e^{-\frac{xu}{u}} du dv = \Pi(\alpha-1) \int_0^\infty v^{\beta-1} \cdot e^{-v} \cdot \psi(1-\alpha, xv) dv$$

$$+\Pi(-\alpha-1)x^{\alpha} \int_0^\infty v^{\alpha+\beta-1} \cdot e^{-v} \psi(1+\alpha, xv) dv,$$

the integrations with respect to the variable v is easily executed; for,

$$\int_0^\infty v^{\beta-1} e^{-v} \psi(1-\alpha, xv) dv = \Pi(\beta-1)\psi(\beta, 1-\alpha, x)$$

$$\int_0^\infty v^{\alpha+\beta-1} e^{-v} \psi(1-\alpha, xv) dv = \Pi(\alpha+\beta-1)\phi(\alpha+\beta, 1+\alpha, x),$$

$$\int_0^\infty v^{\beta-1} \cdot e^{-v} \cdot e^{-\frac{xv}{u}} dv = \frac{\Pi(\beta-1)}{\left(1+\frac{x}{u}\right)^{\beta}}$$

whence

$$\int_0^\infty \int_0^\infty u^{\alpha-1} \cdot e^{-u} \cdot v^{\beta-1} \cdot e^{-v} du dv = \Pi(\beta-1) \int_0^\infty \frac{u^{\alpha-1} e^{-u} du}{(1+\frac{x}{u})^\beta}$$

which integral, writing ux instead of u, is changed into

$$\Pi(\beta-1)x^\alpha \int_0^\infty \frac{u^{\alpha+\beta-1} \cdot e^{-ux} du}{(1+u)^\beta},$$

having substituted which we have

$$\Pi(\beta-1)x^\alpha \int_0^\infty \frac{u^{\alpha+\beta-1} \cdot e^{-ux} du}{(1+u)^\beta}$$

$$= \Pi(\alpha-1)\Pi(\beta-1)\phi(\beta, 1-\alpha, x) + \Pi(-\alpha-1)\Pi(\alpha+\beta-1)x^\alpha \phi(\alpha+\beta, 1+\alpha, x),$$

which formula, changing α into $\alpha - \beta$, is reduced to this more convenient form

$$13. \quad \int_0^\infty \frac{u^{\alpha-1} \cdot e^{-ux} du}{(1+u)^\beta}$$

$$= \frac{\Pi(\alpha-\beta-1)}{\Pi(\alpha-1)} x^\beta \cdot \phi(\beta, \beta-\alpha+1, x) + \frac{\Pi(\beta-\alpha-1)}{\Pi(\beta-1)} x^\alpha \cdot \phi(\alpha, \alpha-\beta+1, x).$$

Since the one side the equation, having interchanged the quantities α and β, remains the same, it has to be

$$14. \quad \frac{x^\alpha}{\Pi(\alpha-1)} \int_0^\infty \frac{u^{\alpha-1} \cdot e^{-ux} du}{(1+u)^\beta} = \frac{x^\beta}{\Pi(\beta-1)} \int_0^\infty \frac{u^{\beta-1} \cdot e^{-ux} du}{(1+u)^\beta}.$$

If the transformation, equation (7.) contains, is applied to formula (13.), we find

$$15. \quad \frac{x^\alpha}{\Pi(\alpha-1)} \int_0^\infty \frac{u^{\beta-1} \cdot e^{-ux} du}{(1+u)^\beta} = \chi(\alpha, \beta, x).$$

Since the series $\chi(\alpha, \beta, x)$ extends to the class of semiconvergent series, it seems necessary to confirm formula (15.) by an own proof, from which it becomes clear at the same time, that by computation of a certain number of initial terms of the series an approximate value of the integral is found. For this purpose I apply the known equation

$$1-\frac{\beta}{1}z+\frac{\beta(\beta+1)}{1\cdot 2}z^2-\cdots+(-1)^{k-1}\frac{\beta(\beta+1)\cdots(\beta+k-2)}{1\cdot 2\cdots(k-1)}z^{k-1}$$

$$=\frac{1}{(1+z)^{\beta}}-\frac{(-1)^k\beta(\beta+1)\cdots(\beta+k-1)}{1\cdot 2\cdot 3\cdots k}z^k\int_0^1\frac{(1-u)^{k-1}du}{(1+zu)^{\beta+k}};$$

by putting $z=\frac{v}{x}$, multiplying by $v^{\alpha-1}\cdot e^{-v}\cdot dv$, then integrating from $v=0$ to $v=\infty$ and dividing $\Pi(\alpha-1)$ we find

$$16.\quad 1-\frac{\alpha\cdot\beta}{1\cdot x}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot x^2}-\cdots(-1)^{k-1}\frac{\alpha(\alpha-1)\cdots(\alpha+k-2)\beta(\beta+1)\cdots(\beta+k-2)}{1\cdot 2\cdot\cdots(k-1)\cdot x^{k-1}}$$

$$=\frac{1}{\Pi(\alpha-1)}\int\frac{v^{\alpha-1}\cdot e^{-v}\cdot dv}{(1+\frac{v}{x})^{\beta}}-\frac{(-1)^k\beta(\beta+1)\cdots(\beta+k-1)}{\Pi(\alpha-1)1\cdot 2\cdot 3\cdots(k-1)x^k}\int_0^1\int_0^{\infty}\frac{(1-u)^{k-1}\cdot v^{\alpha+k-1}\cdot e^{-v}\cdot dv\cdot du}{(1+\frac{uv}{x})^{\beta+k}}$$

this double integral together with its coefficient indicates the error, which is committed, if the integral

$$\frac{1}{\Pi(\alpha-1)}\int\frac{v^{\alpha-1}\cdot e^{-v}\cdot dv}{(1+\frac{v}{x})^{\beta}},\quad\text{or what it the same}\quad\frac{x^{\alpha}}{\Pi(\alpha-1)}\int\frac{v^{\alpha-1}\cdot e^{-vx}\cdot dv}{(1+v)^{\beta}}$$

is approximated by the first k terms of the series of that series. If k is so large that $\beta+k$ is positive, the quantity, which we called the error, changes its sign as k changes into $k+1$, or, if a certain number of terms of that series is computed, this sum is either larger or smaller than the integral in question, but if the subsequent term of the series is added, this new sum is smaller than the integral in question, if that one was larger, and it is larger, if that one was smaller. Therefore, the sums, which that series yields are alternatively too large and too small, and it is clear that an approximate value is found, if the computation is extended to the smallest terms of the semicovergent series. The same can be proven from equation (16.) this way. Obviously, for positive $\beta+k$:

$$\int_0^1\int_0^{\infty}\frac{(1-u)^{k-1}\cdot v^{\alpha+k-1}\cdot e^{-v}dvdu}{(1+\frac{uv}{x})^{\beta+k}}<\int_0^1\int_0^{\infty}(1-u)^{k-1}\cdot v^{\alpha-k-1}\cdot e^{-v}\cdot dvdu$$

and

$$\int_0^1 \int_0^\infty (1-u)^{k-1} \cdot e^{-v} \cdot v^{\alpha+k-1} dv du = \frac{\Pi(\alpha+k-1)}{k},$$

therefore, the error, which is expressed by that double integral, is always smaller than

$$\frac{\beta(\beta+1)\cdots(\beta+k-1)\Pi(\alpha+k-1)}{1\cdot 2\cdot 3\cdots k\cdot \Pi(\alpha)x^k},$$

since which is the first neglected term, it follows that the error is always smaller than that term of the series, to which the summation is extended.
Having put $\beta = 1-\alpha$ equation (15.) goes over into this one:

$$\frac{x^\alpha}{\Pi(\alpha-1)} \int_0^\infty (u+u^2)^{\alpha-1} \cdot e^{-ux} \cdot du$$
$$= \frac{\Pi(2\alpha-2)}{\Pi(\alpha-1)} x^{1-\alpha} \cdot \phi(1-\alpha, 2-2\alpha, x) + \frac{\Pi(-2\alpha)}{\Pi(-\alpha)} x^\alpha \cdot \phi(\alpha, 2\alpha, x),$$

having transformed which series according to formula (6.), we have

$$\frac{x^\alpha}{\Pi(\alpha-1)} \int_0^\infty (u+u^2)^{\alpha-1} \cdot e^{-ux} \cdot du$$
$$= \frac{\Pi(2\alpha-2)}{\Pi(\alpha-1)} x^{1-\alpha} \cdot e^{\frac{x}{2}} \cdot \psi\left(\frac{3}{2}-\alpha, \frac{x^2}{16}\right) + \frac{\Pi(-2\alpha)}{\Pi(-\alpha)} x^\alpha \cdot e^{\frac{x}{2}} \cdot \psi\left(\frac{1}{2}+\alpha, \frac{x^2}{16}\right);$$

further, if x is changed into $4\sqrt{x}$, α into $\alpha+\frac{1}{2}$, by a few reductions we have

$$17. \quad \frac{2^{2\alpha+1} \cdot \sqrt{\pi} \cdot x^\alpha \cdot e^{-2\sqrt{x}}}{\Pi\left(\alpha-\frac{1}{2}\right)} \int_0^\infty (u+u^2)^{\alpha-\frac{1}{2}} \cdot e^{-4u\sqrt{x}} \cdot du$$
$$= \Pi(\alpha-1)\psi(1-\alpha, x) + \Pi(-\alpha-1)x^\alpha \psi(1+\alpha, x),$$

hence by comparison to formula (12.) it follows

$$\int_0^\infty u^{\alpha-1} \cdot e^{-u} \cdot e^{-\frac{x}{u}} \cdot du = \frac{2^{2\alpha+1} \cdot \sqrt{\pi} \cdot x^\alpha \cdot e^{-2\sqrt{x}}}{\Pi\left(\alpha-\frac{1}{2}\right)} \int_0^\infty (u+u^2)^{\alpha-\frac{1}{2}} \cdot e^{-4u\sqrt{x}} \cdot du,$$

from this formula, or if you like it more, from formula (12.), having put $\alpha = \frac{1}{2}$, one easily deduces the extraordinarily simple value of the integral

$$18. \quad \int_0^\infty e^{-u^2} \cdot e^{-\frac{x}{u^2}} \cdot du = \frac{\sqrt{\pi}}{2} \cdot e^{-2\sqrt{x}}.$$

The integrals we just found have many applications in analysis, e. g., in the integration of the Riccati equation, which by simple substitutions can be changed into the form of (9.); but I will not spend more time on these integrals here, but want to answer questions on similar other integrals, of which I first take this one:

$$19. \quad z = \int_0^{\frac{\pi}{2}} \cos^{v-1} v \cdot \cos\left(\frac{1}{2}x \tan v + \beta v\right).$$

I assume the quantity x to be positive all the time, since its negative sign can be transferred to the quantity β. By differentiation of the quantity

$$\cos^{\alpha-1} v \cdot \sin\left(\frac{1}{2}x \tan v + \beta v\right)$$

we find

$$d\left(\cos^{\alpha-1} v \sin\left(\frac{1}{2}x \tan v + \beta v\right)\right) = -(\alpha-1)\cos^{\alpha-1} v \cdot \sin v \cdot \sin\left(\frac{1}{2}x \tan v + \beta v\right) dv$$
$$+\left(\frac{x}{2\cos^2 v} + \beta\right)\cos^{\alpha-1} v \cos\left(\frac{1}{2}x \tan v + \beta v\right),$$

and by integrating from $v = 0$ to $v = \frac{\pi}{2}$

$$20. \quad 0 = -(\alpha-1)\int_0^{\frac{\pi}{2}} \cos^{\alpha-2} \cdot \sin v \cdot \sin\left(\frac{1}{2}x \tan v + \beta v\right) dv$$
$$+\frac{x}{2}\int_0^{\frac{\pi}{2}} \cos^{\alpha-2} \cdot \cos v \cdot \sin\left(\frac{1}{2}x \tan v + \beta v\right) dv$$
$$+\beta\int_0^{\frac{\pi}{2}} \cos^{\alpha-1} \cdot \cos v \cdot \sin\left(\frac{1}{2}x \tan v + \beta v\right) dv$$

further,

$$\frac{dz}{dx} = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{\alpha-2} \cdot \sin v \cdot \sin\left(\frac{1}{2}x\tan v + \beta v\right) dv,$$

$$\frac{dz^2}{dx^2} = -\frac{1}{4}\int_0^{\frac{\pi}{2}} \cos^{\alpha-3} \cdot \sin^2 v \cdot \cos\left(\frac{1}{2}x\tan v + \beta v\right) dv,$$

and hence

$$z - 4\frac{d^2z}{dx^2} = \int_0^{\frac{\pi}{2}} \cos^{\alpha-3} v \cos\left(\frac{1}{2}x\tan v + \beta v\right) dv,$$

having substituted which, equation (30.) goes over into this one

$$21. \quad 0 = (x + 2\beta)z + 4(\alpha - 1)\frac{dz}{dx} - 4x\frac{d^2z}{dx^2},$$

this equation, by means of the substitution $z = e^{-\frac{x}{2}}y$, is transformed into this one

$$0 = \frac{\beta - \alpha + 1}{2}y + (\alpha - 1 + x)\frac{dy}{dx} - x\frac{d^2y}{dx^2},$$

whose complete integral is:

$$y = A\phi\left(\frac{\beta - \alpha + 1}{2}, 1 - \alpha, x\right) + Bx^\alpha\phi\left(\frac{\beta + \alpha + 1}{2}, 1 + \alpha, x\right),$$

and since $z = e^{-\frac{x}{2}} \cdot y$, we have

$$22. \quad \int_0^{\frac{\pi}{2}} \cos^{\alpha-1} v \cdot \cos\left(\frac{1}{2}x\tan v + \beta v\right) dv$$

$$= A \cdot \phi\left(\frac{\beta - \alpha - 1}{2}, 1 - \alpha, x\right) + Bx^\alpha\phi\left(\frac{\beta + \alpha + 1}{2}, 1 + \alpha, x\right).$$

The determination of the constant A is easily obtained by putting $x = \infty$, if α is a positive quantity, but the determination of the other constant requires peculiar artifices: We will obtain both constants at the same time by this method. Multiply equation (22.) by $x^{\lambda-1}e^{-\frac{x}{2}}dx$ and integrate from the limit $x = 0$ to $x = \infty$, having done what

$$23.\quad \int_0^\infty \int_0^{\frac{\pi}{2}} \cos^{\alpha-1} v \cdot x^{\lambda-1} e^{-\frac{x}{2}} \cos\left(\frac{1}{2} x \tan v + \beta\right) dv dx$$

$$= A \int_0^\infty x^{\lambda-1} \cdot e^{-x} \phi\left(\frac{\beta-\alpha+1}{2}, 1-\alpha, x\right) dx$$

$$+ B \int_0^\infty x^{\lambda+\alpha-1} \cdot e^{-x} \phi\left(\frac{\beta+\alpha+1}{2}, 1+\alpha, x\right) dx$$

The values of all these integrals can be expressed by known functions, for

$$\int_0^\infty x^{c-1} \cdot e^{-x} \cdot \phi(a, b, x) dx = \Pi(c-1) F(c, a, b, 1),$$

where F denotes the known hypergeometric series, having expressed which by the function Π,

$$\int_0^\infty x^{c-1} \cdot e^{-x} \cdot \phi(a, b, x) dx = \frac{\Pi(c-1)\Pi(b-1)\Pi(b-a-c-1)}{\Pi(b-a-1)\Pi(b-c-1)},$$

further,

$$\int_0^\infty x^{\lambda-1} \cdot e^{-\frac{x}{2}} \cdot \cos\left(\frac{1}{2} x \tan v + \beta v\right) dx = 2^\lambda \Pi(\lambda-1) \cos^\lambda v \cdot \cos(\lambda+\beta) v,$$

whence that double integral goes over into this one

$$2^\lambda \Pi(\lambda-1) \int_0^{\frac{\pi}{2}} \cos^{\alpha+\lambda-1} \cdot \cos(\lambda+\beta) v \cdot dv,$$

whose value is expressed by the function Π this way

$$\frac{\pi \cdot \Pi(\lambda-1)\, \Pi(\alpha+\lambda-1)}{2^\alpha \Pi\left(\frac{\alpha-\beta-1}{2}\right) \Pi\left(\frac{\alpha-\beta-1}{2}+\lambda\right)}$$

having substituted which, equation (23.) goes over into this one:

$$\frac{\pi\cdot\Pi\left(\lambda-1\right)\Pi\left(\alpha+\lambda-1\right)}{2^{\alpha}\Pi\left(\frac{\alpha-\beta-1}{2}\right)\Pi\left(\frac{\alpha-\beta-1}{2}+\lambda\right)}$$

$$=A\frac{\Pi(\lambda-1)\Pi(-\alpha)\Pi\left(-\frac{\alpha+\beta+1}{2}-\lambda\right)}{\Pi\left(-\frac{\alpha+\beta+1}{2}\right)\Pi(-\alpha-\lambda)}+B\frac{\Pi(\alpha+\lambda-1)\Pi(\alpha)\Pi\left(-\frac{\alpha+\beta+1}{2}-\lambda\right)}{\Pi\left(\frac{\alpha-\beta-1}{2}\right)\Pi(-\lambda)},$$

this equation is easily transformed to this more convenient form

$$\frac{\pi\cdot\cos\left(\frac{\alpha+\beta}{2}+\lambda\right)\pi}{2^{\alpha}\Pi\left(\frac{\alpha-\beta-1}{2}\right)}=\frac{A\cdot\Pi(-\alpha)\sin(\alpha+\lambda)\pi}{\Pi\left(-\frac{\alpha+\beta+1}{2}\right)}+\frac{B\cdot\Pi(\alpha)\sin\lambda\pi}{\Pi\left(\frac{\alpha-\beta-1}{2}\right)},$$

which, since it has to hold for every arbitrary value of the quantity λ, yields these two

$$\frac{\pi\cos\frac{\alpha+\beta}{2}\pi}{2^{\alpha}\Pi\left(\frac{\alpha-\beta-1}{2}\right)}=\frac{A\cdot\sin(\alpha\pi)\Pi(-\alpha)}{\Pi\left(-\frac{\alpha+\beta+1}{2}\right)},$$

$$-\frac{\pi\sin\frac{\alpha+\beta}{2}\pi}{2^{\alpha}\Pi\left(\frac{\alpha-\beta-1}{2}\right)}=\frac{A\cdot\cos(\alpha\pi)\Pi(-\alpha)}{\Pi\left(-\frac{\alpha+\beta+1}{2}\right)}+\frac{B\cdot\Pi(\alpha)}{\Pi\left(\frac{\alpha-\beta-1}{2}\right)},$$

from which one easily finds the values of the constants A and B

$$A=\frac{\pi\cdot\Pi(\alpha-1)}{2^{\alpha}\Pi\left(\frac{\alpha-\beta-1}{2}\right)\Pi\left(\frac{\alpha+\beta-1}{2}\right)},\quad B=-\frac{\pi\cdot\cos\left(\frac{\alpha-\beta}{2}\right)\pi}{2^{\alpha}\cdot\sin\alpha\pi\Pi(\alpha)},$$

having finally substituted which values in equation (22.)

$$24.\quad\int\limits_0^{\frac{\pi}{2}}\cos^{\alpha-1}v\cdot\cos\left(\frac{1}{2}x\tan v+\beta v\right)dv$$

$$\frac{\pi\cdot\Pi(\alpha-1)e^{-\frac{x}{2}}\cdot\phi\left(\frac{\beta-\alpha+1}{2},1-\alpha,x\right)}{2^{\alpha}\Pi\left(\frac{\alpha-\beta-1}{2}\right)\Pi\left(\frac{\alpha+\beta-1}{2}\right)}-\frac{\pi\cdot\cos\frac{\alpha-\beta}{2}\pi\cdot x^{\alpha}\cdot e^{-\frac{x}{2}}\phi\left(\frac{\beta+\alpha+1}{2},1+\alpha,x\right)}{2^{\alpha}\sin\alpha\pi\Pi(\alpha)}.$$

The most simple special cases of this formula are:

$$25.\quad \int_0^{\frac{\pi}{2}} \cos^{\alpha-1} v \cdot \cos(x \tan v - (\alpha+1)v)dv = \frac{\pi \cdot x^\alpha \cdot e^{-x}}{\Pi(\alpha)},$$

$$26.\quad \int_0^{\frac{\pi}{2}} \cos^{\alpha-1} v \cdot \cos(x \tan v + (\alpha+1)v)dv = 0,$$

the one of which is obtained for $\beta = -\alpha - 1$, the other for $\beta = \alpha + 1$. Combining formulas (25.) and (26.) also these follows

$$27.\quad \int_0^{\frac{\pi}{2}} \cos^{\alpha-1} v \cdot \cos(x \tan v) \cos(\alpha+1)v \cdot dv = \frac{\pi \cdot x^\alpha \cdot e^{-x}}{2\Pi(\alpha)},$$

$$28.\quad \int_0^{\frac{\pi}{2}} \cos^{\alpha-1} v \cdot \sin(x \tan v) \sin(\alpha+1)v \cdot dv = \frac{\pi \cdot x^\alpha \cdot e^{-x}}{2\Pi(\alpha)}.$$

Formulas (25.) and (26.) agree with those found by Laplace, which others later demonstrated by other method, confer Crelle Journal Volume 13 p. 231, where Liouville by the method of differentiation with respect to arbitrary indices found

$$\int_{-\infty}^{\infty} \frac{e^{\alpha\sqrt{-1}} \cdot d\alpha}{(x + \alpha\sqrt{-1})^\mu} = \frac{2\pi \cdot e^{-x}}{\Gamma(\mu)}.$$

Having put $\beta = \alpha - 1$ formula (24.) yields another very simple integral

$$29.\quad \int_0^{\frac{\pi}{2}} \cos^{\alpha-1} v \cdot \cos(x \tan v + (\alpha-1)v)dv = \frac{\pi e^{-x}}{2^\alpha}.$$

The two series, which are contained in the one side of equation (24.), having put $\beta = 0$, become $\phi\left(\frac{1-\alpha}{2}, 1-\alpha, x\right)$ and $\phi\left(\frac{1+\alpha}{2}, 1+\alpha, x\right)$, and they can be transformed into series of the class ψ by means of formula (6.). Having done these transformations, if α is changed into 2α, x into $4\sqrt{x}$, the following formula results

$$10.\quad \frac{2\Pi\left(\alpha - \frac{1}{2}\right)}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \cos^{2\alpha-1} v \cdot \cos(2\sqrt{x} \tan v)dv$$

$$= \Pi(\alpha-1)\psi(1-\alpha, x) + \Pi(-\alpha-1) \cdot x^\alpha \cdot \psi(1+\alpha, x),$$

hence by comparison with formula (12.)

$$31.\quad \frac{2\Pi\left(\alpha-\frac{1}{2}\right)}{\sqrt{\pi}}\int\limits_0^{\frac{\pi}{2}}\cos^{2\alpha-1}v\cdot\cos(2\sqrt{x}\tan v)dv=\int\limits_0^{\infty}u^{\alpha-1}\cdot e^{-u}\cdot e^{-\frac{x}{u}}\cdot du.$$

In like manner one can demonstrate the connection of the two integrals, which are contained in the equations (13.) and (24.); for, this formula (24.), if one writes $\alpha-\beta$ instead α, $\alpha+\beta-1$ instead of β and multiplies by $\frac{1}{\pi}\Pi(-\beta)\cdot 2^{\alpha}\cdot e^{\frac{x}{2}}\cdot x^{\beta}$, takes on the form

$$32.\quad \frac{2\Pi(-\beta)\cdot e^{\frac{x}{2}}\cdot x^{\beta}}{\pi}\int\limits_0^{\frac{\pi}{2}}(2\cos v)^{\alpha-\beta-1}\cdot\cos\left(\frac{1}{2}x\tan v+(\alpha+\beta-1)v\right)dv$$

$$=\frac{\Pi(\alpha-\beta-1)}{\Pi(\alpha-1)}x^{\beta}\phi(\beta,\beta-\alpha+1,x)+\frac{\Pi(\beta-\alpha-1)}{\Pi(\beta-1)}x^{\alpha}\phi(\alpha,\alpha-\beta+1,x),$$

having compared which with formula (13.), one sees that

$$33.\quad \int\limits_0^{\infty}\frac{u^{\beta-1}\cdot e^{-ux}\cdot du}{(1+u)^{\alpha}}$$

$$=\frac{2\cdot e^{\frac{x}{2}}}{\sin\beta\pi}\int\limits_0^{\frac{\pi}{2}}(2\cos v)^{\alpha-\beta-1}\cdot\cos\left(\frac{1}{2}x\tan v+(\alpha+\beta-1)v\right)dv,$$

furthermore, if the one side of equation (32.) is transformed by formula (7.),

$$34.\quad \frac{2\Pi(-\beta)e^{\frac{x}{2}}\cdot x^{\beta}}{\pi}\int\limits_0^{\frac{\pi}{2}}(2\cos v)^{\alpha-\beta-1}\cdot\cos\left(\frac{1}{2}x\tan v+(\alpha+\beta-1)v\right)dv=\chi(\alpha,\beta,x).$$

In like manner we will also treat the more general integral

$$y=\int\limits_0^{\frac{\pi}{2}}\sin^{\alpha-1}v\cdot\cos^{\beta-1}v\cdot\cos(x\tan v+\gamma v)dv$$

and we will find the cases, in which it can be expressed by the series mentioned above. We always assume the quantity x to be positive in this integral, since its negative sign can be transferred to the quantity γ. By differentiating formula $\sin^\alpha v \cdot \cos^\beta v \cdot \cos(x \tan v + \gamma v)$, then integrating from $u = 0$ to $u = \frac{\pi}{2}$, we find

$$\begin{aligned}
0 = \alpha \int_0^{\frac{\pi}{2}} & \sin^{\alpha-1} v \cdot \cos^{\beta+1} v \cdot \cos(x \tan v + \gamma v) dv \\
- \beta \int_0^{\frac{\pi}{2}} & \sin^{\alpha+1} v \cdot \cos^{\beta-1} v \cdot \cos(x \tan v + \gamma v) dv \\
- x \int_0^{\frac{\pi}{2}} & \sin^{\alpha} v \cdot \sin^{\beta-2} v \cdot \cos(x \tan v + \gamma v) dv \\
- \gamma \int_0^{\frac{\pi}{2}} & \sin^{\alpha} v \cdot \cos^{\beta} v \cdot \sin(x \tan v + \gamma v) dv,
\end{aligned}$$

from this equation, if the integrals are expressed by y and its differentials, one easily deduces this differential equation of third order:

$$35. \quad 0 = \alpha y + (\gamma + x)\frac{dy}{dx} + (\beta - 2)\frac{d^2y}{dx^2} - x\frac{d^3y}{dx^3},$$

now, if one puts

$$y = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots$$

one easily finds conditional equations, which must hold among the coefficients of this series for it to satisfy the differential equation:

$$\begin{aligned}
\alpha A_0 + \gamma \cdot 1 \cdot A_1 - 1 \cdot 2 \cdot (2 - \beta) A_2 \\
(\alpha + 1) A_1 + \gamma \cdot 2 \cdot A_2 - 2 \cdot 3 \cdot (3 - \beta) A_3,
\end{aligned}$$

and generally

$$37. \quad (\alpha + k) A_k + \gamma \cdot (k+1) A_{k+1} - (k+1)(k+2)(k+2-\beta) A_{k+2}.$$

The same way, if one sets

$$38. \quad y = x^\beta (B_0 + B_1 x + B_2 x^2 + B_3 x^3 + \cdots),$$

one finds these relations among the coefficients

$$\gamma \cdot \beta B_0 - \beta(\beta+1)\cdot 1 \cdot B_1,$$
$$(\alpha+\beta)B_0 + \gamma(\beta+1)B_1 - (\beta+1)(\beta+2)\cdot 2 \cdot B_2,$$

and generally

$$39. \quad (\alpha+\beta+k)B_k + \gamma(\beta+k+1)B_{k+1} - (\beta+k+1)(\beta+k+2)(k+2)B_{k+2},$$

hence it is plain that the complete integral of equation (35.) is

$$40. \quad y = A_0 + A_1 x + A_2 x + \cdots + x^\beta (B_0 + B_1 x + B_2 x^2 \cdots),$$

for, by equations (37.) two of the quantities A_0, A_1, A_2 etc., and by the equations (39.) one of the quantities B_0, B_1, B_2 etc. remains arbitrary, so that this integral contains three arbitrary constants. Therefore, if one substitutes the integral propounded above for y again,

$$41. \quad \int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v \cdot \cos^{\beta-1} v \cdot \cos(x \tan v + \gamma v) dv$$
$$= A_0 + A_1 x + A_2 x^2 + \cdots + x^\beta (B_0 + B_1 x + B_2 x^2 + \cdots).$$

From the relations among the coefficients it is easily seen that these series and this integral are higher transcendents than those we want to consider here; but nevertheless in certain cases they agree with those. First, if we assume $\gamma = \alpha + \beta$, from the equations (39.) it follows

$$\begin{aligned} B_1 &= \frac{\alpha+\beta}{1(1+\beta)} B_0, \\ B_2 &= \frac{(\alpha+\beta)(\alpha+\beta+1)}{1\cdot 2\cdot (1+\beta)(2+\beta)} B_0, \\ B_3 &= \frac{(\alpha+\beta)(\alpha+\beta+1)(\alpha+\beta+2)}{1\cdot 2\cdot 3(1+\beta)(2+\beta)(3+\beta)} B_0 \\ \text{etc.} \quad & \text{etc.} \end{aligned}$$

Further, if β is positive, having put $x = 0$, it follows from equation (41.)

$$A_0 = \int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v \cdot \cos^{\beta-1} v \cdot \cos(\alpha+\beta)v \cdot dv = \frac{\cos\frac{\alpha\pi}{2}\Pi(\alpha-1)\Pi(\beta-1)}{\Pi(\alpha+\beta-1)},$$

the same way, if equation (41.) is differentiated with respect to x and one sets $x = 0$ after this, we have

$$A_1 = -\int_0^{\frac{\pi}{2}} \sin^{\alpha} v \cdot \cos^{\beta-2} v \cdot \sin(\alpha+\beta)v \cdot dv = -\frac{\cos\frac{\alpha\pi}{2}\Pi(\alpha)\Pi(\beta-2)}{\Pi(\alpha+\beta-1)}.$$

therefore,

$$A_1 = \frac{\alpha}{1(1-\beta)} A_0,$$

hence from the equations (37.) it easily follows

$$\begin{aligned} A_2 &= \frac{\alpha(\alpha+1)A_0}{1\cdot 2(1-\beta)(2-\beta)}, \\ A_1 &= \frac{\alpha(\alpha+1)(\alpha+2)A_0}{1\cdot 2\cdot 3(1-\beta)(2-\beta)(3-\beta)} \\ &\text{etc.} \quad \text{etc.} \end{aligned}$$

Therefore, in this case, in which $\gamma = \alpha + \beta$, the two series, by means of which we expressed our integral, extends to that class of series, which we denoted by ϕ above, and formula (41.) goes over into this one:

$$\int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v \cdot \cos^{\beta-1} v \cdot \cos(x\tan v + (\alpha+\beta)v)dv$$
$$= \frac{\cos\frac{\alpha\pi}{2}\Pi(\alpha-1)\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}\phi(\alpha, 1-\beta, x) + B_0 x^{\beta}\phi(\alpha+\beta, 1+\beta, x).$$

For the determination of the constant B_0 we will use the same method as above for the determination of the constants of (22.). Multiplying by $x^{\lambda-1} \cdot e^{-x} \cdot dx$ and integrating from 0 to ∞

$$\Pi(\lambda-1)\int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v \cdot \cos^{\beta+\lambda-1} v \cdot \cos(\alpha+\beta+\lambda) v dv$$

$$= \frac{\cos\frac{\alpha\pi}{2}\Pi(\alpha-1)\Pi(\beta-1)\Pi(\lambda-1)}{\Pi(\alpha+\beta-1)} F(\lambda,\alpha,1-\beta,1)$$

$$+B_0\Pi(\beta+\lambda-1)F(\lambda+\beta,\alpha+\beta,1+\beta,1),$$

and having expressed the hypergeometric series together with the integral by the function Π,

$$\frac{\cos\frac{\alpha\pi}{2}\Pi(\lambda-1)\Pi(\alpha-1)\Pi(\beta+\lambda-1)}{\Pi(\alpha+\beta+\lambda-1)}$$

$$= \frac{\cos\frac{\alpha\pi}{2}\Pi(\lambda-1)\Pi(\alpha-1)\Pi(\beta-1)\Pi(-\beta)\Pi(-\beta-\alpha-\lambda)}{\Pi(\alpha+\beta-1)\Pi(-\alpha-\beta)\Pi(-\beta-\lambda)}$$

$$B_0\frac{\Pi(\beta+\lambda-1)\Pi(\beta)\Pi(-\beta-\alpha-\lambda)}{\Pi(-\alpha)\Pi(-\beta)}$$

after some reductions the quantity, as it has to, vanishes completely, and the very simple value of the constant B_0 results

$$B_0 = \cos\frac{\alpha\pi}{2}\Pi(-\beta-1),$$

having finally substituted which, we have

$$42. \quad \int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v \cdot \cos^{\beta-1} v \cdot \cos(x\tan v + (\alpha+\beta)v)$$

$$= \frac{\cos\frac{\alpha\pi}{2}\Pi(\alpha-1)\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}\phi(\alpha,1-\beta,x)$$

$$+x^\beta\cos\frac{\alpha\pi}{2}\Pi(-\beta-1)\phi(\alpha+\beta,1+\beta,x).$$

A similar formula is deduced from this one by changing α into $\alpha-1$, β into $\beta+1$ and differentiating

$$42. \quad \int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v \cdot \cos^{\beta-1} v \cdot \sin(x\tan v + (\alpha+\beta)v)$$

$$= \frac{\sin\frac{\alpha\pi}{2}\Pi(\alpha-1)\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}\phi(\alpha, 1-\beta, x)$$

$$+x^\beta \sin\frac{\alpha\pi}{2}\Pi(-\beta-1)\phi(\alpha+\beta, 1+\beta, x).$$

and having compared these formulas to each other, one sees the connection of the two integrals

$$43. \qquad \cos\frac{\alpha\pi}{2}\int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v\cdot\cos^{\beta-1} v\cdot\sin(x\tan v+(\alpha+\beta)v)dv$$

$$= \sin\frac{\alpha\pi}{2}\int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v\cdot\cos^{\beta-1} v\cdot\cos(x\tan v+(\alpha+\beta)v)dv$$

which formula can also be exhibited this way

$$44. \quad \int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v\cdot\cos^{\beta-1} v\cdot\sin\left(x\tan v+(\alpha+\beta)v-\frac{\alpha\pi}{2}\right)dv = 0.$$

The special case $\alpha = 0$ of the formula (42.) is remarkable

$$45. \quad \int_0^{\frac{\pi}{2}} \frac{\cos^{\beta-1}\cdot\sin(x\tan v+\beta v)}{\sin v}dv = \frac{\pi}{2},$$

a special case of which, corresponding to the value $X = 0$ Liouville found in this Journal Volume 13 p. 232. Furthermore, from the comparison of formulas (42.) and (13.) one sees the connection of this integral to those we treated above without any difficulty

$$46. \quad \frac{\cos\frac{\alpha\pi}{2}\Pi(\alpha-1)}{\Pi(\alpha+\beta+1)}x^\beta\int_0^\infty \frac{u^{\alpha+\beta-1}\cdot e^{-ux}du}{(1+u)^\alpha}$$

$$= \int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v\cdot\cos^{\beta-1} v\cdot\cos(x\tan v+(\alpha+\beta)v)dv.$$

Another case, in which the series of formula (41.) reduces to the series denoted by the character ϕ, is $\gamma = -\alpha-\beta$; for, in this case the same way as above it is easily found that formula (41.) goes over into this one:

$$\int_0^{\frac{\pi}{2}} \sin^{\alpha-1}\cos^{\beta-1} v \cdot \cos(x\tan v - (\alpha+\beta)v)dv$$

$$= \frac{\cos\frac{\alpha\pi}{2}\Pi(\alpha-1)\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}\phi(\alpha, 1-\beta, -x) + B_0 x^\beta \phi(\alpha+\beta, 1+\beta, -x),$$

but in this case the constant B_0 has another value we will find by multiplying by $x^{\alpha+\beta} \cdot e^{-x}dx$ and by integrating from $x = 0$ to $x = \infty$; after the integrations:

$$\Pi(\alpha+\beta+1)\int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v \cdot \cos^{\alpha+2\beta-1} v \cdot dv$$

$$= \cos\frac{\alpha\pi}{2}\Pi(\alpha-2)\Pi(\beta-1)F(\alpha+\beta, \alpha, 1-\beta, -1)$$
$$+B_0\Pi(\alpha+2\beta-1)F(\alpha+2\beta, \alpha+\beta, 1+\beta, -1),$$

also these hypergeometric series, whose fourth element $= -1$, can be expressed in terms of the function Π according to the formula

$$F(\alpha, \beta, \alpha-\beta+1, -1) = \frac{2^{-\alpha}\sqrt{\pi}}{\Pi\left(\frac{\alpha}{2}-\beta\right)\Pi\left(\frac{\alpha-1}{2}\right)},$$

which I demonstrated in the paper on the hypergeometric series of this journal Volume 15 p. 135. Hence, if that integral and hypergeometric series are expressed in terms of the function Π, after some simple reductions it results:

$$B_0 = \cos\left(\frac{\alpha}{2}+\beta\right)\pi\ \Pi(-\beta-1),$$

and having substituted the value of this constant:

$$47. \quad \int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v \cdot \cos^{\beta-1} v \cdot \cos(x\tan v - (\alpha+\beta)v)dv$$

$$= \frac{\cos\frac{\alpha\pi}{2}\Pi(\alpha-1)\Pi(\beta-1)}{\Pi(\alpha+\beta-1)}\phi(\alpha, 1-\beta, -x)$$
$$+x^\beta\cos\left(\frac{\alpha}{2}+\beta\right)\pi\Pi(-\beta-1)\phi(\alpha+\beta, 1+\beta, -x).$$

A similar formula is easily deduced from this one by changing α into $\alpha - 1$, β into $\beta + 1$ and differentiating with respect to the variable x

$$48. \quad \int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v \cdot \cos^{\beta-1} v \cdot \sin(x \tan v - (\alpha+\beta)v) dv$$

$$= -\frac{\sin \frac{\alpha\pi}{2} \Pi(\alpha-1)\Pi(\beta-1)}{\Pi(\alpha+\beta-1)} \phi(\alpha, 1-\beta, -x)$$

$$-x^\beta \sin\left(\frac{\alpha}{2}+\beta\right) \pi \Pi(-\beta-1)\phi(\alpha+\beta, 1+\beta, -x).$$

These formulas (47.) and (48.) can easily be combined in two ways so that they have these simpler forms:

$$49. \quad \int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v \cdot \cos^{\beta-1} v \cdot \sin\left(x \tan v - (\alpha+\beta)v + \left(\frac{\alpha}{2}+\beta\right)\pi\right) dv$$

$$= \frac{\pi \Pi(\alpha-1)\phi(\alpha, 1-\beta, -x))}{\Pi(-\beta)\Pi(\alpha+\beta-1)},$$

$$50. \quad \int_0^{\frac{\pi}{2}} \sin^{\alpha-1} v \cdot \cos^{\beta-1} v \cdot \sin\left(x \tan v - (\alpha+\beta)v + \frac{\alpha\pi}{2}+\right) dv$$

$$= \frac{\pi x^\beta}{\Pi(\beta)} \phi(\alpha+\beta, 1+\beta, -x)$$

In all integrals treated here, as we mentioned above already, x must always be a positive quantity, but if x would be assumed to be negative, all sums we found would be false; in this regard the integral of equation (50.) is especially remarkable, since for positive x is equal to that series, but vanishes for negative x, confer equation (44.).

Written in Liegnitz, in the month of April, 1837